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CAUSAL FIR MATRICES WITH ANTICAUSAL FIR INVERSES, AND APPLICATION IN CHARACTERIZATION OF BIORTHONORMAL FILTER BANKS

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Abstract

Causal FIR matrices with anticausal FIR inverses have a key role in the theory of FIR perfect reconstruction filter banks. We explore the theory of such matrices. Some general results on nature of inverses of first order causal FIR matrices are then presented. This leads, in particular, to a complete parameterization of the *Biorthonormal Lapped Transform* (*BOLT*) reported earlier in [1]. [1]

1. INTRODUCTION

It is well-known that the M-channel maximally decimated analysis/synthesis system of Fig. 1.1(a) can be redrawn as in Fig. 1.1(b) where $E(z)$ and $R(z)$ are the polyphase matrices of the analysis and synthesis bank respectively. In a recent paper [1] we argued that the set of all FIR perfect reconstruction (PR) systems can be essentially characterized by characterizing all *CAusal Fir transfer matrices with AntiCAusal Fir Inverses*, abbreviated *cafacafi*. In particular, this gave rise to the so-called *BiOrthnormal Lapped Transform*, abbreviated *BOLT*. Some theorems pertaining to the factorization of *cafacafi* systems and *BOLT* systems, as well as the most general degree one *cafacafi* building block were presented in [1], along with examples of unfactorizable *cafacafi* systems. After a brief review of the results in [1], we present new results for first order polyphase matrices, and establish a complete factorization of the *BOLT* (which was missing in [1]).

1.1. Importance of cafacafi systems

The filter bank in Fig. 1.1 is a perfect reconstruction (PR) system [i.e., $\hat{x}(n) = x(n)$] if and only if $R(z) = E^{-1}(z)$. It is an FIR filter bank if both of the matrices $E(z)$ and $R(z)$ are FIR. So, for an FIR PR system we should have $[\det E(z)] = cz^{-J}, c \neq 0$. In fact all FIR PR systems are characterized by this property. Given an arbitrary FIR PR system, if we define $E_1(z) = z^{-I}E(z)$ and $R_1(z) = z^{I}R(z)$ for arbitrary I, we have constructed a new FIR PR filter bank.

The new analysis and synthesis filters are $z^{-IM}H_k(z)$ and $z^{IM}F_k(z)$, which means that the frequency characteristics are unchanged. By choosing I large enough we can make $E_1(z)$ causal and its inverse $R_1(z)$ anticausal. So the class of all FIR PR filter banks can be characterized by considering the *cafacafi* class. Unlike arbitrary FIR systems with FIR inverses, *cafacafi* systems have special system-theoretic properties [2],[3], and generally speaking, are more interesting. In [3] it is shown that the most general degree one *cafacafi* system has the form $G_0V_m(z)$ where

$$V_m(z) = I - u_m v_m^\dagger + z^{-1}u_m v_m^\dagger, \qquad (1.1)$$

G_0 is nonsingular, and u_m and v_m are $M \times 1$ vectors with $u_m^\dagger v_m = 1$. (Superscript $\dagger$ stands for transpose conjugate). With $u_m = v_m$, $V_m(z)$ is the familiar degree-one paraunitary (PU) building block [4],[5]. It is clear that $V_m(1) = I$; it can be shown that $[\det V_m(z)] = z^{-1}$. Inspite of the generality of this form, an arbitrary *cafacafi* system may not be factorizable into these building blocks! An example is $V(z) = uv^\dagger + z^{-1}I + z^{-2}uv^\dagger, u^\dagger v = 0$, see [3].

1.2. The BOLT

A *BOLT* is a first order causal FIR system

$$G(z) = g(0) + z^{-1}g(1), \qquad (1.2)$$

with anticausal FIR inverse. In short, a *BOLT* is a $M \times M$ first order *cafacafi* system. Its degree is clearly equal to the rank of $g(1)$ [5].

Recall that the lapped orthogonal transform (*LOT*) [6], [7], is essentially a filter bank in which the polyphase matrix $E(z)$ is paraunitary with order one. This means that $E(z)$ has the form (1.2), and its inverse is $g^\dagger(0) + zg^\dagger(1)$ which is FIR anticausal! So the *BOLT* is a generalization of the *LOT*; the *LOT* is paraunitary, whereas the *BOLT* merely *cafacafi*.

It turns out that, while arbitrary *cafacafi* systems cannot be factorized, any *BOLT* can be factorized into degree one building blocks:

$$G(z) = G(1)V_\rho(z)V_{\rho-1}(z)\dots V_1(z) \qquad (1.3)$$

where $V_m(z)$ are as in (1.1) with $u_m^\dagger v_m = 1$ (Fig. 1.2). This therefore is a generalization of the paraunitary factorization [4] for the first order *cafacafi* case. The PU factorization, however holds for any order.

Given that each building block $V_m(z)$ is *cafacafi*, it is easily verified that the product (1.3) is *cafacafi*. However, the product in general is not a *BOLT*, that is it may not be a first order system as in (1.2). This, however, can be ensured by imposing further conditions on u_i and v_i. This result (Sec. 3) will follow directly from the more general results on first order systems to be developed in the next section.

2. INVERSES OF FIRST ORDER SYSTEMS

Consider a first order $M \times M$ transfer matrix as in (1.2). We will assume that $G(1) = G(e^{j0})$ is nonsingular. This is the case, for example, when there exists an FIR inverse. We can then express $G(z) = G(1)F(z)$ where $F(1) = I$. So we can write $F(z) = I - P + z^{-1}P$ without loss of generality. Let ρ be the rank of P [i.e., ρ = degree of $F(z)$] so that we can write

$$F(z) = I - \mathcal{U}\mathcal{V}^\dagger + z^{-1}\mathcal{U}\mathcal{V}^\dagger \qquad (2.1)$$

where $\mathcal{U}$ and $\mathcal{V}$ are $M \times \rho$. The type of inverse that $F(z)$ has depends strongly on the matrix $\mathcal{V}^\dagger\mathcal{U}$, in particular on its eigenvalues. We now present some precise results on this. Throughout this section $F(z)$ is the $M \times M$ system shown above; $\mathcal{U}$ and $\mathcal{V}$ are $M \times \rho$ with rank ρ.

Lemma 2.1. $F(z)$ has an anticausal inverse if and only if $\mathcal{V}^\dagger\mathcal{U}$ is nonsingular. ◇

Proof. A minimal implementation of $F(z)$ is shown in Fig. 2.1. The state space description of this structure (with standard notations [5]) is given by $A = 0$, $B = \mathcal{V}^\dagger$, $C = \mathcal{U}$, and $D = I - \mathcal{U}\mathcal{V}^\dagger$. Let

$$\mathcal{R} \triangleq \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 0 & \mathcal{V}^\dagger \\ \mathcal{U} & I - \mathcal{U}\mathcal{V}^\dagger \end{bmatrix} \qquad (2.2)$$

As shown in [2], $\mathcal{R}$ is nonsingular if and only if there exists an anticausal inverse for $F(z)$. We will show that $\mathcal{R}$ is nonsingular if and only if $\mathcal{V}^\dagger\mathcal{U}$ is nonsingular. Suppose $\mathcal{R}x = 0$ for some vector x. Then

$$\mathcal{V}^\dagger x_2 = 0, \quad \text{and} \quad \mathcal{U}x_1 + x_2 = 0, \quad \text{with} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Combining these we get $\mathcal{V}^\dagger\mathcal{U}x_1 = 0$. If $\mathcal{V}^\dagger\mathcal{U}$ is nonsingular then $x_1 = 0$ and so $x_2 = -\mathcal{U}x_1 = 0$ from above. In short, if $\mathcal{R}x = 0$ then $x = 0$. So $\mathcal{R}$ is nonsingular. On the other hand, if $\mathcal{V}^\dagger\mathcal{U}$ is singular there exists $y \neq 0$ such that $\mathcal{V}^\dagger\mathcal{U}y = 0$. If we now choose $x_1 = -y$ and $x_2 = \mathcal{U}y$, then $\mathcal{R}x = 0$ so $\mathcal{R}$ singular. ▽▽▽

Theorem 2.1. The inverse of $F(z)$ is

1. FIR if and only if all eigenvalues of $\mathcal{U}\mathcal{V}^\dagger$ are restricted to be 0's and 1's.

2. FIR and anticausal (i.e., $F(z)$ is *cafacafi*) if and only if $\mathcal{U}\mathcal{V}^\dagger$ has ρ of its eigenvalues equal to unity and the remaining $M - \rho$ eigenvalues equal to zero.

3. FIR and causal (i.e., $F(z)$ is unimodular in z^{-1}) if and only if $\mathcal{U}\mathcal{V}^\dagger$ has all eigenvalues = 0. ◇

Proof. From the unitary triangularization theorem [8] we can write $\mathcal{U}\mathcal{V}^\dagger = T\Delta T^\dagger$ where $TT^\dagger = I$, and Δ is upper triangular with the eigenvaules $\{\lambda_0, \lambda_1 \ldots \lambda_{\rho-1}, 0, \ldots 0\}$ on the diagonals. (Since the rank is ρ there could be at most ρ nonzero eigenvalues). We can then express $F(z) = T(I - \Delta + z^{-1}\Delta)T^\dagger$ so that $[\det F(z)] = \prod_{i=0}^{\rho-1}(1 - \lambda_i + z^{-1}\lambda_i)$. This has the form cz^{-K} (which is necessary and sufficient for the existence of FIR inverse) if and only if $\lambda_i = 0$ or 1 for each i. Since deg $F(z) = \rho$ the FIR inverse is anticausal if and only if $\det F(z) = cz^{-\rho}$ (Theorem 5.3, [2]). This will be the case if and only if $\mathcal{U}\mathcal{V}^\dagger$ has ρ eigenvalues equal to unity (and, of course, the remaining $M - \rho$ eigenvalues = 0). Finally the FIR inverse is causal (i.e., $F(z)$ unimodular) if and only if the determinant is constant, i.e., $\lambda_i = 0$ for all i. ▽▽▽

Note. Since $\mathcal{U}\mathcal{V}^\dagger$ has rank ρ it can have at most ρ nonzero eigenvalues. But there could be fewer, as in the extreme example of a triangular matrix with all diagonal elements equal to zero.

Theorem 2.2. The inverse of the system $F(z)$ depends on the properties of $\mathcal{V}^\dagger\mathcal{U}$ as follows.

1. $F(z)$ has an anticausal inverse if and only if $\mathcal{V}^\dagger\mathcal{U}$ is nonsingular.

2. The inverse of $F(z)$ is FIR if and only if all eigenvalues of $\mathcal{V}^\dagger\mathcal{U}$ are restricted to be 0's and 1's.

3. The inverse of $F(z)$ is FIR and anticausal (i.e., $F(z)$ is *cafacafi*) if and only if $\mathcal{V}^\dagger\mathcal{U}$ has all eigenvalues equal to unity.

4. The inverse of $F(z)$ is FIR and causal (i.e., $F(z)$ is unimodular) if and only if $\mathcal{V}^\dagger\mathcal{U}$ has all eigenvalues equal to zero. ◇

Proof. Part 1 is a repetition of Lemma 2.1. Parts 2 and 4 follow from Theorem 2.1 by using the fact that every nonzero eigenvalue of the matrix PQ is an eigenvalue of QP (for any P, Q for which PQ and QP are defined). Part 3 follows by combining parts 1 and 2; indeed, nonsingularity of $\mathcal{V}^\dagger\mathcal{U}$ and the condition that the eigenvalues be restricted to be ones and zeros is equivalent to the statement that all the eigenvalues of $\mathcal{V}^\dagger\mathcal{U}$ are equal to unity. ▽▽▽

Example 2.1. The cases where $\mathcal{V}^\dagger\mathcal{U} = I_\rho$ and $\mathcal{V}^\dagger\mathcal{U} = 0$ give examples of FIR systems with anticausal and

causal FIR inverses respectively. With $\mathbf{F}(z) = (\mathbf{I} - \mathcal{U}\mathcal{V}^\dagger + z^{-1}\mathcal{U}\mathcal{V}^\dagger)$ we have

$$\mathbf{F}^{-1}(z) = \begin{cases} \mathbf{I} - \mathcal{U}\mathcal{V}^\dagger + z\mathcal{U}\mathcal{V}^\dagger & \text{for } \mathcal{V}^\dagger\mathcal{U} = \mathbf{I}_\rho \\ \mathbf{I} + \mathcal{U}\mathcal{V}^\dagger - z^{-1}\mathcal{U}\mathcal{V}^\dagger & \text{for } \mathcal{V}^\dagger\mathcal{U} = 0 \end{cases}$$

as one can verify by direct multiplication. Notice that if $\mathcal{V}^\dagger\mathcal{U} = \mathbf{I}_\rho$ then the inverse is also of first order. So first order *cafacafi* systems with *higher order* inverses [3] are not covered by this example. Eq. (1.1) is a special case of $\mathbf{F}(z)$ where $\rho = 1$.

Example 2.2. The following example

$$\mathcal{U}\mathcal{V}^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathcal{V}^\dagger\mathcal{U} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

satisfies part 4 of Theorem 2.2 so that $\mathbf{F}(z)$ is unimodular, even though $\mathcal{V}^\dagger\mathcal{U} \neq 0$.

Example 2.3. As a special case consider $\mathbf{I} - \mathbf{P} + z^{-1}\mathbf{P}$ where $\mathbf{P}^2 = \mathbf{P}$. With ρ denoting the rank of $\mathbf{P}$, we can write $\mathbf{P} = \mathcal{U}\mathcal{V}^\dagger$. Now $\mathbf{P}^2 = \mathbf{P}$ implies $\mathcal{U}\mathcal{V}^\dagger\mathcal{U}\mathcal{V}^\dagger = \mathcal{U}\mathcal{V}^\dagger$. Premultiplying by $\mathcal{U}^\dagger$ and postmultiplying with $\mathcal{V}$ and using the facts that $\mathcal{U}^\dagger\mathcal{U}$ and $\mathcal{V}^\dagger\mathcal{V}$ are nonsingular we obtain $\mathcal{V}^\dagger\mathcal{U} = \mathbf{I}$. From part 3 of Theorem 2.2 we therefore conclude that there exists an anticausal FIR inverse for $\mathbf{I} - \mathbf{P} + z^{-1}\mathbf{P}$, when $\mathbf{P}^2 = \mathbf{P}$. In fact the inverse is $\mathbf{I} - \mathbf{P} + z\mathbf{P}$, as verified by substitution.

Example 2.4. Unimodular system. By a slight modification of the above theorem we can show that $\mathbf{I} + z^{-1}\mathcal{U}\mathcal{V}^\dagger$ is unimodular if and only if $\mathcal{V}^\dagger\mathcal{U}$ has all eigenvalues equal to zero.

3. COMPLETE PARAMETERIZATION OF BOLT

The *BOLT* matrix has the form (1.2) and has anticausal FIR inverse. Since this implies $\mathbf{G}(1)$ is nonsingular, we can write $\mathbf{G}(z) = \mathbf{G}(1)\mathbf{F}(z)$ where $\mathbf{F}(z)$ is as in (2.1). Using Theorem 2.2 (part 3) we can say that a system $\mathbf{G}(z)$ is *BOLT* if and only if it has the form

$$\mathbf{G}(z) = \mathbf{G}(1)(\mathbf{I} - \mathcal{U}\mathcal{V}^\dagger + z^{-1}\mathcal{U}\mathcal{V}^\dagger) \tag{3.1}$$

where $\mathcal{V}^\dagger\mathcal{U}$ has all eigenvalues equal to unity. In Sec. 1.2 we saw that the *BOLT* can be factorized as in (1.3) where $V_m(z)$ is as in (1.1) with $\mathbf{v}_m^\dagger\mathbf{u}_m = 1$ and ρ is the degree of $\mathbf{G}(z)$. Conversely an arbitrary product of the form (1.3) with $V_m(z)$ as above, still represents a system with anticausal FIR inverse, but may have order > 1 i.e., it may not be a *BOLT*. Suppose we impose the further restriction that

$$\mathbf{v}_k^\dagger\mathbf{u}_i = \begin{cases} 0, & 1 \le i \le k-1, \\ 1, & i = k \end{cases} \tag{3.2}$$

Then it can be verified that (1.3) reduces to (3.1) with the constant matrices $\mathcal{V}$ and $\mathcal{U}$ given by

$$\mathcal{V} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_\rho \end{bmatrix}, \quad \mathcal{U} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_\rho \end{bmatrix}. \tag{3.3}$$

Conversely, can we represent any *BOLT* system as in (1.3) with the restriction (3.2)? The answer is in the affirmative: if $\mathbf{G}(z)$ is *BOLT*, this means in particular that it has an FIR inverse, and so $\mathbf{G}(1)$ is nonsingular. So we can always write a degree ρ *BOLT* as in (3.1), where $\mathcal{U}$ and $\mathcal{V}$ are $M \times \rho$ matrices with rank ρ. Now $\mathcal{U}\mathcal{V}^\dagger = \mathcal{U}\mathbf{T}\mathbf{T}^\dagger\mathcal{V}^\dagger$ for any unitary $\mathbf{T}$, and we can rewrite $\mathcal{U}\mathcal{V}^\dagger = \mathcal{U}_1\mathcal{V}_1^\dagger$ by defining $\mathcal{U}_1 = \mathcal{U}\mathbf{T}$ and $\mathcal{V}_1^\dagger = \mathbf{T}^\dagger\mathcal{V}^\dagger$. Note that $\mathcal{V}_1^\dagger\mathcal{U}_1 = \mathbf{T}^\dagger\mathcal{V}^\dagger\mathcal{U}\mathbf{T}$. By proper choice of $\mathbf{T}$ we can ensure that $\mathcal{V}_1^\dagger\mathcal{U}_1$ is a triangular matrix. In other words, we can assume without loss of generality that $\mathcal{V}^\dagger\mathcal{U}$ is triangular. Since $\mathbf{G}(z)$ is *cafacafi* the matrix $\mathcal{V}^\dagger\mathcal{U}$ has all eigenvalues equal to unity (part 3, Theorem 2.2), so

$$\mathcal{V}^\dagger\mathcal{U} = \begin{bmatrix} 1 & \times & \times & \cdots & \times \\ 0 & 1 & \times & \cdots & \times \\ 0 & 0 & 1 & \cdots & \times \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \tag{3.4}$$

where $\times$ stands for possibly nonzero elements. Now denote the columns of $\mathcal{V}$ and $\mathcal{U}$ as in (3.3). Then the property (3.4) means that (3.2) is satisfied.

Thus we have defined $\mathbf{u}_m$ and $\mathbf{v}_m$, $1 \le m \le \rho$ satisfying (3.2). We already know that if such $\mathbf{u}_m$ and $\mathbf{v}_m$ are used in (1.3), the result has the form (3.1) with $\mathcal{U}$ and $\mathcal{V}$ given by (3.3). So the given BOLT matrix (3.1) can indeed be represented as in (1.3), with the vectors satisfying (3.2). Summarizing, we have:

Theorem 3.1. *BOLT Characterization.* Consider an $M \times M$ transfer matrix $\mathbf{G}(z)$. We say that this is a *BOLT* if $\mathbf{G}(z) = \mathbf{g}(0) + z^{-1}\mathbf{g}(1)$, and it has an anticausal FIR inverse. The following statements are equivalent:

1. $\mathbf{G}(z)$ is a *BOLT*.

2. $\mathbf{G}(z)$ can be factorized as in (1.3) where $\mathbf{G}(1)$ is nonsingular and $V_m(z)$ are as in (1.1), with the vectors $\mathbf{v}_k$ and $\mathbf{u}_i$ satisfying (3.2).

3. $\mathbf{G}(z)$ can be written in the form $\mathbf{G}(z) = \mathbf{G}(1)(\mathbf{I} - \mathcal{U}\mathcal{V}^\dagger + z^{-1}\mathcal{U}\mathcal{V}^\dagger)$, where $\mathbf{G}(1)$ is nonsingular and $\mathcal{V}^\dagger\mathcal{U}$ has all eigenvalues equal to unity.

4. $\mathbf{G}(z)$ can be written in the form $\mathbf{G}(z) = \mathbf{G}(1)(\mathbf{I} - \mathcal{U}\mathcal{V}^\dagger + z^{-1}\mathcal{U}\mathcal{V}^\dagger)$, where $\mathbf{G}(1)$ is nonsingular and $\mathcal{V}^\dagger\mathcal{U}$ has the triangular form (3.4).

Thus if $\mathbf{G}(z)$ is *BOLT* with $\mathcal{V}^\dagger\mathcal{U}$ written in the form (3.4), the columns of $\mathcal{V}$ and $\mathcal{U}$ satisfy (3.2), and can be taken to be the vectors $\mathbf{v}_m$ and $\mathbf{u}_m$ in the factorization (1.3). So the factorization is determined simply by identifying the columns of $\mathcal{V}$ and $\mathcal{U}$. ◊

Degrees of freedom. The nonsingular matrix $\mathbf{G}(1)$ has M^2 elements, and the vectors $\mathbf{u}_k, \mathbf{v}_k$, have

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M elements each, but constrained by (3.2). In the real coefficient case it can be verified that the $M \times M$ degree-ρ $BOLT$ has $M^2 + 2\rho M - 0.5\rho(\rho + 1)$ degrees of freedom. This should be compared with the LOT which has $0.5M(M - 1) + \rho M - 0.5\rho(\rho + 1)$ degrees of freedom. Traditional transform coding (special case with $\mathcal{U}\mathcal{V}^\dagger = 0$ and $G(1)$ unitary) has $0.5M(M - 1)$ freedoms. The extra freedom offered by the $BOLT$ can perhaps be exploited to obtain better attenuation for the analysis filters, or to impose other constraints such as linear phase, regularity (for wavelet synthesis [9]) and so forth. This requires detailed investigation. Fig. 3.1 shows the magntiude responses of the 8-channel analysis filters for the $BOLT$ and the LOT, with the same degree $\rho = 4$.
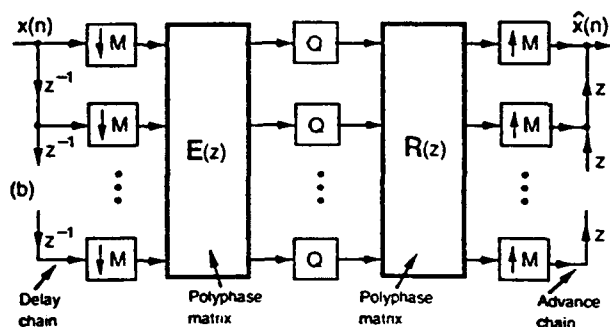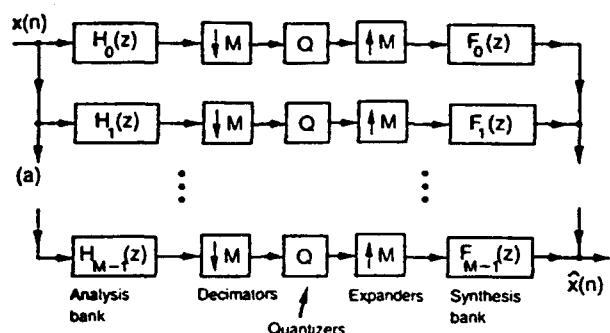
Fig. 1.1 (a) The maximally decimated filter bank, and (b) the polyphase representation.
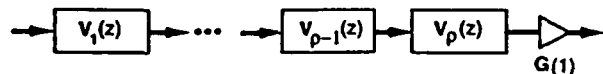

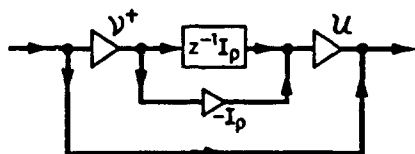Fig. 1.2. The cascaded structure for the BOLT.

Fig. 2.1. A minimal implementation of (2.1).

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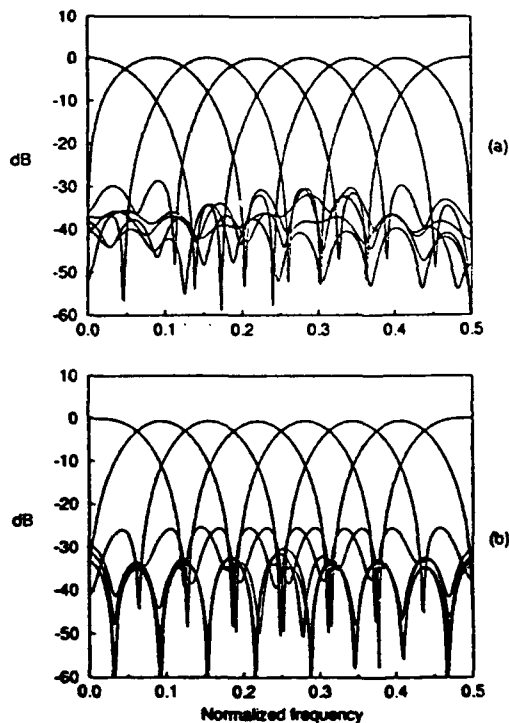[9] Daubechies, I. Ten lectures on wavelets, SIAM, CBMS series, April 1992.

Fig. 3.1. Magnitude responses of the analysis filters for (a) the BOLT and (b) the LOT.

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